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RISING SEEPAGE OF A HEAVY LIQUID IN A VERTICAL CYLINDER†

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A formulation and solution of the problem of steady seepage in a vertical column when the liquid enters from below are given. The boundary conditions on the free surface are linearized by the transposition method. Expressions are obtained for the dispersion surface and the components of the seepage rate and pressure. A numerical analysis of the solution is carried out. © 2002 Elsevier Science Ltd. All rights reserved.

The problem considered is associated with the process of water separation which is used in the hydrotransport of suspensions.

We note the following solutions of problems of seepage with a free surface: the solution of seepage problems with an unknown boundary by the method of the theory of functions of a complex variable for plane motions in homogeneous dams of comparatively simple configuration [1–3], the solution of the problem of the spreading of a heap of ground waters of arbitrary initial shape by linearizing the conditions on the depression surface [4], and the solution of the spatial problem of the spreading of a heap, in particular, in the shape of a parallelepiped [5].

1. FORMULATION OF THE PROBLEM

The flow scheme and the system of coordinates are shown in Fig. 1. The origin of a cylindrical system of coordinates is placed at the centre of the cross-section, the r axis lies in a plane which passes through the beginning of the perforated segment of the water separation section and the z axis is directed vertically upwards. The cylindrical grid of the water separation section is shown by the dashed lines.

The following assumptions are made. The liquid is viscous and incompressible. The flow is laminar and isothermal. The rate of ascent of the porous skeleton is homogeneous throughout the cross-section and equal to v_-, that is, in essence a porous rod is considered. In particular, the skeleton can also be fixed. The permeability is homogeneous throughout the volume. An atmospheric pressure (p_+) acts on the lateral seepage surface (of radius R). Evaporation and capillary effects on the free surface are ignored.

In the cross-section $z = 0$ in a grid of thickness δ, there is a pressure drop of the liquid from the value at the input of the water separation section from p_0 to p_+. We will assume that there is a linear pressure distribution throughout the thickness of the grid. It is also assumed, without any substantial disturbance of the flow pattern, that the above-mentioned pressure drop occurs on a section of the porous skeleton which is immediately adjacent to the grid, as shown in Fig. 1. If it is assumed that there is a stepwise change in the pressure in the grid ($r = R, z = 0, \partial r \to \infty$), then the radial velocity will be infinite at the point $z = 0, r = R$, which contradicts the real flow pattern. An analysis showed that the flow field barely changes when δ/R is varied from 0.001 to 0.1. The hydraulic resistance of the grid in the seepage section was not taken into consideration. The effect of a grid on seepage was considered previously in [6].

Suppose the equation of the free surface is $z = l - f(r)$, where l is the maximum ascent of the liquid (the height of the steady heap) and $f(r)$ is an unknown function ($\sup(f) \ll l$). On the depression surface, the pressure in the liquid is equal to atmospheric pressure and, furthermore, the normal component of the velocity is equal to zero.

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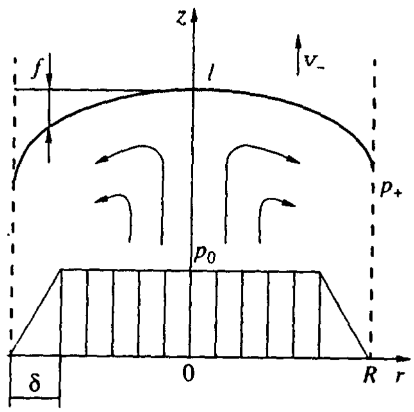

Fig. 1

The components of the seepage rate are determined by the Darcy–Gersevanov equation [7]

$$v_z = -\frac{K}{\eta}\left(\frac{\partial p}{\partial z} + \rho g\right) + v_-, \quad v_r = -\frac{K}{\eta}\frac{\partial p}{\partial r} \tag{1.1}$$

In the case of this equation, the relations between the components of the seepage rate on the free surface

$$z = l - f(r), \quad v_z + v_r \frac{df}{dr} = 0$$

can be written as

$$\frac{\partial p}{\partial z} + A^* + \frac{df}{dr}\frac{\partial p}{\partial r} = 0, \quad A^* = \rho g - \frac{\eta v_-}{K}$$

where η and ρ are the viscosity and density of the liquid and K is the permeability of the porous skeleton.

On substituting expressions (1.1) into the continuity equation, we obtain the Laplace equation for the pressure. Since the function p is sought to within an arbitrary constant, it is possible to put $p_+ = 0$.

We now introduce the dimensionless variables and parameters

$$\zeta = \frac{z}{R}, \quad \xi = \frac{r}{R}, \quad P = \frac{p}{p_0}, \quad F = \frac{f}{R}, \quad A = \frac{A^* R}{P_0}, \quad L = \frac{l}{R}, \quad \Delta = \frac{\delta}{R}$$

$$M = \frac{A^* \pi R^2 K}{q\eta}$$

where q is the liquid flow rate, and we write the problem in the dimensionless form

$$\frac{1}{\xi}\frac{\partial P}{\partial \xi} + \frac{\partial^2 P}{\partial \xi^2} + \frac{\partial^2 P}{\partial \zeta^2} = 0, \quad 0 < \xi < 1, \quad 0 < \zeta < L - F(\xi) \tag{1.2}$$

$$\zeta = 0: \quad P = \begin{cases} 1, & 0 \leqslant \xi \leqslant 1 - \Delta \\ (1-\xi)\Delta^{-1}, & 1 - \Delta \leqslant \xi \leqslant 1 \end{cases} \tag{1.3}$$

$$\xi = 1: \quad P = 0; \qquad \xi = 0: \quad P < \infty \tag{1.4}$$

$$\zeta = L - F(\xi): \quad P = 0, \quad \frac{\partial P}{\partial \zeta} + A + \frac{\partial F}{\partial \xi}\frac{\partial P}{\partial \xi} = 0 \tag{1.5}$$

$$\xi = L, \quad \xi = 0: \quad F = 0 \tag{1.6}$$

The problem is an analogy of the Dirichlet problem but differs in the form of the boundary conditions.

2. SOLUTION OF THE PROBLEM

We will linearize the problem by transposing boundary conditions (1.5) from the surface $\zeta = L - F(\xi)$ to the horizontal plane $\zeta = L$ using a Taylor expansion of the function P and its derivatives in the neighbourhood of the point $\xi = L$. On substituting these expansions into boundary conditions (1.5), we obtain

$$P(\xi, L) - F\frac{\partial P(\xi, L)}{\partial \zeta} = 0 \tag{2.1}$$

$$\frac{\partial P(\xi, L)}{\partial \zeta} - F\frac{\partial^2 P(\xi, L)}{\partial \zeta^2} + A + \frac{dF}{d\xi}\left[\frac{\partial P(\xi, L)}{\partial \xi} - F\frac{\partial^2 P(\xi, L)}{\partial \xi \partial \zeta}\right] = 0 \tag{2.2}$$

In this way, boundary conditions (1.5) are transposed from the surface $\zeta = L - F(\xi)$ to the plane $\zeta = L$ and the function F is eliminated from the arguments of P.

The solution of Laplace's equation (1.2), taking (1.3) and (1.4) into account, has the form

$$P = \sum_k [A_k \operatorname{ch}(\mu_k \zeta) + B_k \operatorname{sh}(\mu_k \zeta)] J_0(\mu_k \xi), \quad A_k = \frac{2\varphi_k}{\mu_k J_1(\mu_k)} \tag{2.3}$$

$$\varphi_k = \frac{1}{\Delta}\left\{1 - (1-\Delta)^2 \frac{J_1[\mu_k(1-\Delta)]}{J_1(\mu_1)} - \frac{\mu_k}{J_1(\mu_k)} \int_{1-\Delta}^{1} \xi^2 J_0(\mu_k \xi) d\xi\right\}$$

Henceforth, the summation over k and n is carried out from unity to infinity, and μ_k, μ_n are the roots of the equation $J_0(\mu) = 0$.

Note the following properties of the coefficients φ_k: $\varphi_k \to 1$ when $\Delta \to 0$, $\varphi_k < 1$ when $0 < \Delta < 1$.

We will now represent the function F and the constant A in relation (2.2) in the form of a Fourier–Bessel series

$$F = \sum_k a_k J_0(\mu_k \xi), \quad A = 2A \sum_k \frac{J_0(\mu_k \xi)}{\mu_k J_1(\mu_k)} \tag{2.4}$$

The last expansion results from the fact that there is no zero mode in (2.3).

The unknown coefficients B_k in (2.3) and a_k in (2.4) are found from conditions (2.1) and (2.2). On considering relations (2.1), (2.3) and (2.4) together, we obtain

$$\sum_k (A_k C_k + B_k S_k) J_0(\mu_k \xi) - \sum_n a_n J_0(\mu_n \xi) \sum_k \mu_k (A_k S_k + B_k C_k) J_0(\mu_k \xi) = 0 \tag{2.5}$$

$$C_k = \operatorname{ch}(\mu_k L), \quad S_k = \operatorname{sh}(\mu_k L)$$

where a_n are the coefficients of the expansion of the function F in a Fourier–Bessel series.

We multiply all the terms of Eq. (2.5) by $\xi d\xi$ and integrate within limits from $\xi = 0$ to $\xi = 1$. Taking account of the mutual orthogonality of the eigenfunctions $J_0(\mu_k \xi)$, we obtain the system of algebraic equations

$$2(A_k C_k + B_k S_k) J_1(\mu_k) - a_k \mu_k^2 J_k^2(\mu_k)(A_k S_k + B_k C_k) = 0, \quad k = 1, 2, \ldots$$

whence we find

$$B_k = A_k \frac{S_k \mu_k^2 a_k J_1(\mu_k) - 2C_k}{2S_k - C_k \mu_k^2 a_k J_1(\mu_k)} \tag{2.6}$$

Substituting expressions (2.3) and (2.4) into condition (2.2), we obtain

$$\sum_k (A_kS_k + B_kC_k)\mu_kJ_0(\mu_k\xi) - \sum_n a_nJ_0(\mu_n\xi)\sum_k (A_kC_k + B_kS_k)\mu_kJ_0(\mu_k\xi) +$$
$$+2A\sum_k J_0(\mu_k\xi)[\mu_kJ_1(\mu_k\xi)]^{-1} + \sum_n a_n(-\mu_n)J_1(\mu_n\xi)\sum_k (A_kC_k + B_kS_k)(-\mu_k)J_0(\mu_k\xi) = 0$$

Multiplying all the terms of this expression by $\xi d\xi$ and integrating from $\xi = 0$ to $\xi = 1$, we arrive at the infinite system of equations

$$(A_kS_k + B_kC_k)\mu_k^2J_1(\mu_k) + 2A = 0, \quad k = 1, 2, \ldots \tag{2.7}$$

From a consideration of relation (2.6) and (2.7) together we find

$$a_k = \frac{2S_k}{C_k\mu_k^2J_1(\mu_k)} - \frac{A_k}{AC_k}, \quad k = 1, 2, \ldots \tag{2.8}$$

Consequently, the function

$$F(\xi) = \frac{2}{A}\sum_k \frac{AS_k - \mu_k\varphi_k}{C_k\mu_k^2J_1(\mu_k)}J_0(\mu_k\xi) \tag{2.9}$$

describes the shape of the depression surface.

According to relation (2.9), the depression surface is horizontal at the top of the heap $r = 0$, since $dF/d\xi = 0$ when $\xi = 0$. The configuration of the surface depends on the parameter A and the initial pressure distribution, which is characterized by the coefficients φ_k.

When account is taken of relations (2.6) and (2.8), the expression for the pressure (2.3) takes the form

$$P = 2\sum_k \{\varphi_k\mu_k \operatorname{ch}[\mu_k(L - \zeta)] - A\operatorname{sh}(\mu_k\zeta)\}\frac{J_0(\mu_k\xi)}{C_k\mu_k^2J_1(\mu_k)} \tag{2.10}$$
$$0 \leqslant \zeta \leqslant L - F(\xi), \quad 0 \leqslant \xi \leqslant 1$$

Using expression (1.1), we find the components of the seepage rate.

The liquid flow rate q and the height l to which it rises depend on the pressure at the inlet p_0. We therefore find the relation between the parameters A, M and L.

Since $F(\xi = 1) = 0$, the height of the seepage surface is equal to l. Hence, the liquid flow rate can be defined by the integral

$$q = 2\pi R\int_0^l v_r(r = R)dz$$

When account is taken of relations (1.2) and (2.10), after some reduction we obtain

$$M = A\left[2\sum_k A_k \operatorname{th}(\mu_kL)J_1(\mu_k) + 4A\sum_k \frac{1 - C_k}{C_k\mu_k^2}\right]^{-1} \tag{2.11}$$

In order to obtain the second equation, we use boundary condition (1.6) for function (2.9) and find the following dependence of A on L

$$A = \sum_k \frac{\varphi_k}{C_k\mu_kJ_1(\mu_k)}\left[\sum_k \frac{S_k}{C_k\mu_k^2J_1(\mu_k)}\right]^{-1} \tag{2.12}$$

Hence, the dependence of the pressure at the inlet p_0 on the height of the heap l is described by expression (2.12) and, correspondingly, the dependence of q on l is described by expression (2.11).

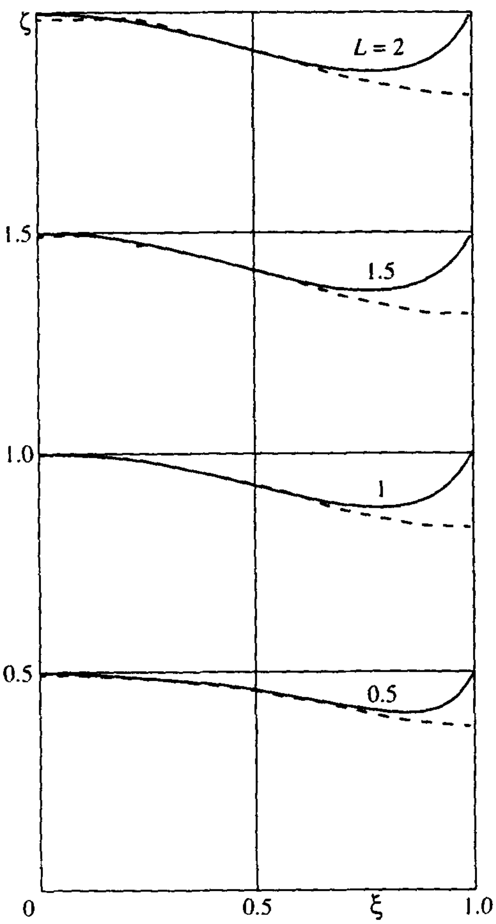

Fig. 2

3. RESULTS OF CALCULATIONS

The lines of the free surfaces were constructed for different values of L for the case when $\Delta = 0.1$. The coefficients φ_k were calculated up to $k = 100$. The integral occurring in the expression for φ_k was calculated using Simpson's formula (it can be represented in the form of a series but, when $\mu_k \geqslant 10$ ($k \geqslant 3$), the series has poor convergence). The parameter A (for a given value of L) was calculated using formula (2.12).

The relatives $\zeta = L - F(\xi)$, calculated using formula (2.9), are represented by the solid curves in Fig. 2, and the same relations, calculated by numerical solution of the equation $P(\zeta, \xi) = 10^{-6}$ in which the quantity P was found using formula (2.10), are represented by the dashed curves. In the latter case, the profile of an isobar was determined for a pressure close to atmospheric pressure. In the neighbourhood of the wall ($\xi = 1$), the free surface is significantly distorted, which is due to the Gibbs effect [9], which is characteristic for Fourier series. Increasing the number of terms in series (2.9) did not remove this distortion. In the domain $\xi \leqslant 0.7$, the isobars practically coincide with the calculated lines of the free surfaces, and no Gibbs effect is observed in the neighbourhood of the wall. As the height to which the liquid rises increases, the shape of the free surface is stabilized.

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